Sum of the cubes of the first n Fibonacci's numbers.
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Calculate the sum of the cubes of the first n Fibonacci's numbers.
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Let (F_n) be sequence of Fibonacci numbers defined by
$F_{n+1} = F_n + F_{n-1}, n \in \mathbb{N}$ and $F_0 = 0, F_1 = 1.$
We will prove that $\sum_{k=1}^{n} F_k^3 = \frac{F_{3n+2} + 6(-1)^{n-1}F_{n-1} + 5}{10}$
By replacing m in **Luca's formula** $F_{n+m} = F_{m+1}F_n + F_mF_{n-1}$ with $2n$ we obtain $F_{3n} = F_{n+2n} = F_{2n+1}F_n + F_{2n}F_{n-1}$. Applying identities $F_{2n+1} = F_{n+1}^2 + F_n^2$ and $F_{2n} = F_n(F_{n+1} + F_{n-1})$ we obtain $F_{3n} = F_n(F_{n+1}^2 + F_n^2) + F_{n-1}F_n(F_{n+1} + F_{n-1}) =$
$F_n^3 + F_n^2F_{n+1} + F_{n-1}^2F_n + F_{n-1}F_nF_{n+1} = F_n^3 + F_n(F_{n+1}^2 - 2F_{n-1}F_{n+1} + F_{n-1}^2) + 3F_{n-1}F_nF_{n+1} =$
$F_n^3 + F_n(F_{n+1} - F_{n-1})^2 + 3F_{n-1}F_nF_{n+1} = 2F_n^3 + 3F_{n-1}F_nF_{n+1}$.
Since $F_{n-1} \bullet F_{n+1} - F_n^2 = (-1)^n \Leftrightarrow F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^nF_n$.
Thus, $F_{3n} = 5F_n^3 + 3(-1)^nF_n \Leftrightarrow F_n^3 = \frac{F_{3n} + 3(-1)^{n+1}F_n}{5}$ and, therefore,
since $\sum_{k=1}^{n} F_k^3 = \frac{1}{5}\left(\sum_{k=1}^{n} F_{3k} + 3\sum_{k=1}^{n}(-1)^{k+1}F_k\right)$ remains to find $\sum_{k=1}^{n} F_{3k}, \sum_{k=1}^{n}(-1)^{k+1}F_k$.
We have $\sum_{k=1}^{n} (-1)^kF_k = -F_1 + \sum_{k=2}^{n} (-1)^k(F_{k-1} + F_{k-2}) =$
$-F_1 + \sum_{k=2}^{n} \left((-1)^{k-2}F_{k-2} - (-1)^{k-1}F_{k-1}\right) = -F_1 + F_0 - (-1)^{n-1}F_{n-1} = -1 + (-1)^nF_{n-1}$
Since* $F_{3(n+1)} = 4F_{3n} + F_{3(n-1)}$ then $\sum_{k=1}^{n} F_{3(k+1)} = 4\sum_{k=1}^{n} F_{3k} + \sum_{k=1}^{n} F_{3(k-1)}$.
Let $S_n := \sum_{k=1}^{n} F_{3k}$. Then $\sum_{k=1}^{n} F_{3(k+1)} = S_n + F_{3n+3} - F_3 = S_n + F_{3n+3} - 2$,
$\sum_{k=1}^{n} F_{3(k-1)} = S_n - F_{3n}$ and $4S_n = \sum_{k=1}^{n} 4F_{3k} = \sum_{k=1}^{n} F_{3(k+1)} - \sum_{k=1}^{n} F_{3(k-1)} =$
$S_n + F_{3n+3} - 2 - S_n + F_{3n} = F_{3n+3} + F_{3n} - 2 = F_{3n+4} - F_{3n+2} + F_{3n} - 2$.
But since* $F_{n+2} = 3F_n - F_{n-2}$ for any $n \in \mathbb{N}$ then $F_{3n+4} - F_{3n+2} + F_{3n} = 2F_{3n+2}$
and, therefore, $\sum_{k=1}^{n} F_{3k} = \frac{1}{2}(F_{3n+2} - 1)$.
Hence, $\sum_{k=1}^{n} F_k^3 = \frac{1}{5}\left(\frac{1}{2}(F_{3n+2} - 1) + 3(-1)^{n-1}F_{n-1}\right) = \frac{F_{3n+2} + 6(-1)^{n-1}F_{n-1} + 5}{10}$.
* For any natural $n \geq 2$ we have $F_{n+2} = F_{n+1} + F_n$, $F_{n+1} = F_n + F_{n-1}$, $F_n = F_{n-1} + F_{n-2}$.
Hence $F_{n+2} = 2F_n + F_{n-1} = 2F_n + \left(F_n - F_{n-2}\right) = 3F_n - F_{n-2}$.
Also $F_{n+3} = F_{n+2} + F_{n+1} = \left(3F_n - F_{n-2}\right) + (F_n + F_{n-1}) = 4F_n + (F_{n-1} - F_{n-2}) =$
$4F_n + F_{n-3}$.